CRANK-NICHOLSON AND DU FORT & FRANKEL HYBRID FINITE DIFFERENCE SCHEMES ARISING FROM OPERATOR SPLITTING FOR SOLVING TWO DIMENSIONAL BURGERS EQUATION

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ABSTRACT

Burgers' equation is a fundamental partial differential equation from fluid mechanics. It occurs in various areas of applied mathematics, such as modeling of fluid dynamics and traffic flow. It relates to the Navier-Stokes equation for incompressible flow with the pressure term removed. Due to the complexity of the Analytic solution, one needs to solve the equation by using numerical methods.

29/07/2015

ABSTRACT CONT…..

In this research we develope the pure Crank-Nicholson (CN) Scheme and Crank-Nicholson-Du Fort & Frankel (CN-DF) method by Operator Splitting. Crank-Nicholson-Du-Fort and Frankel is an hybrid scheme made by combining the Crank-Nicholson and Du-Fort and Frankel schemes which are both unconditionally stable but the Du-fort scheme is explicit while the Crank-Nicholson scheme is implicit.

29/07/2015

ABSTRACT CONT…..

The developed schemes are solved numerically using initially solved solution via Hopf-Cole transformation and separation of variables to generate the initial and boundary conditions. Analysis of the resulting schemes was found to be unconditionally stable. The results of the hybrid scheme are found to compare well with those of the pure Crank-Nicholson.

LITERATURE REVIEW

One of the important Mathematical models of basic flow equation describing unsteady transport problem consisting of a class of time dependent partial differential equations is the two dimensional Burger's equation (Burgers, 1948). The two dimensional Burger's equation occur in physical problems like turbulence, flow through a shock wave traveling in a viscous fluid, sedimentation of particles in fluid suspensions under effect of gravity Coupled non-linear Burgers' equations in two dimension is a special form of incompressible Navier-Stokes equations without the pressure term and the continuity equation (Vineet , Mohammad , Utkarsh , & Sanyasira, 2011)

LITERATURE REVIEW CONT…

The Burgers' equation was first introduced by Bateman (Bateman, 1915), who derived the steady state solution for the one-dimensional equation and was studied in details by Burgers (Burgers, 1948). Analytic solution of the Burgers' equation involves series solutions which converge very slowly for small values of viscosity constant according to Idris (Idris & Ali, 2007). Certain types of boundary value problems can be solved by replacing the differential equation by the corresponding finite difference equation and then solving the latter by a process of iteration. These methods have been used by many mathematicians according to Jain (Jain, 2004). Linearized parabolic equations appear as models in heat flow and gas dynamics. Finite difference solutions of these equations are found by using ordinary discretization, see (Ames, 1994) and (Grffiths & Mitchel , 1980).

LITERATURE REVIEW CONT…

. In the past several years, numerical solutions to one-dimensional Burgers' equations have attracted a lot of attention of the researchers (V. K. Srivastava, M. Tamsir, U. Bhardwaj, & Y. Sanyasiraju, 2011). Many researchers use the coupled two dimensional Burgers' equation and is mentioned in (Ali, 2009) (M. Basto, V. Semiao, & F. Calheiros, 2009), (Beauchamp & Arminjon, 1979), (M. M. Rashidi & E. Erfani, 2009), (V. K. Srivastava, M. Tamsir, U. Bhardwaj, & Y. Sanyasiraju, 2011), (B. Zheng, 2010) amongst others.

LITERATURE REVIEW CONT…

Alternating Direction Implicit Formulation of the Differential Quadrature Method (ADI-DQM) has been used in the past to solve the Burgers equation in two-dimension. The numerical results showed that the ADI-DQM has the higher accuracy and convergence as well as the less computation workload by using few grid points (A.S.J. Al-Saif & Mohammed J., 2012)

LITERATURE REVIEW CONT…

Operator splitting is a powerful method for numerical investigation of complex models. It involves splitting complex problem into a sequence of simpler tasks, that can be called split sub-problems (Yesim, 2010). Espen in his thesis (Espen, 2011) discussed numerical quadratures in one and two dimensions, which was followed by a discussion regarding the differentiation of general operators in Banach spaces.

LITERATURE REVIEW CONT…

In the research (Espen, 2011) investigated the Godunov and Strang method numerically for the viscous Burgers' equation and the KdV equation and presented different numerical methods for the sub-equations from the splitting. They discovered that the Operator splitting methods work well numerically for the two equations. Also in his thesis, Yesim (Yesim, 2010) studied consistency and stability of the operator splitting methods. He concentrated on how to improve the classical operator splitting methods via Zassenhaus product formula.

LITERATURE REVIEW CONT….

Hybrid Schemes with Crank-Nicholson was first introduced by 2009 to solve the 1-D heat equation using operator splitting by modifying it. (Koross, Chepkwony, Oduor, & Omolo, 2009). In their paper they developed hybrid finite difference method resulting from operator splitting for solving the modified form and proved that there is an improvement in efficacy of the Crank-Nicholson scheme when the Lax-Friedrich's and Du Fort and Frankel discretizations are used on it. They concluded in their research findings that the Crank-Nicholson-Lax-Friedrich-Du For and Frankel is the most accurate method for solving 1-D heat equation.

LITERATURE REVIEW CONT….

Du Fort-Frankel can be traced back to 1953 when it was presented as one of the numerical methods for solving the heat equation with periodic boundary conditions, (E.C. Du Fort & S.P. Frankel, 1953) The scheme is explicit, and it is unconditionally stable for the initial value problem. The generalized Du Fort-Frankel scheme has been tested for the Burger's equation using the 4th order accurate operators and the scheme developed was run with $\Delta x = \Delta t = 0.1$, and the error was found to be 16 times smaller in accordance with the 4th order accuracy steady state solution was found with no sign of instability (David Gottlieb & Bertil Gustafsson, 1976). It however faces consistency problems for large values of Δt

INTRODUCTION

Crank-Nicholson Method

Crank–Nicholson method is a finite difference method used for numerically solving the heat equation and similar partial differential equations. It is a second-order method in time, implicit in time, and is numerically stable. The method was developed by John Crank and Phyllis Nicholson in the mid-20th century.

STATEMENT OF THE PROBLEM

We develop hybrid finite difference schemes arising from operator splitting for solving 2-D Burgers equation of the form:

$$\left.\begin{aligned} u_t + uu_x + vu_y &= \frac{1}{\boldsymbol{Re}}(u_{xx} + u_{yy}) \\ v_t + uv_x + vv_y &= \frac{1}{\boldsymbol{Re}}(v_{xx} + v_{yy}) \end{aligned}\right\}$$

STATEMENT OF THE PROBLEM CONT…

Subject to initial conditions:

$$\left.\begin{aligned} u(x,y,0) &= f(x,y), \;\; (x,y) \in D \\ v(x,y,0) &= g(x,y), \;\; (x,y) \in D \end{aligned}\right\}$$

and boundary conditions:

$$\left.\begin{aligned} u(x,y,t) &= f_1(x,y), \;\; x,y \in \partial D, t > 0 \\ v(x,y,t) &= g_1(x,y), \;\; x,y \in \partial D, t > 0 \end{aligned}\right\}$$

Where $\mathrm{D} = \{(\mathrm{x},y)|a \leq x \leq b, a \leq y \leq b\}$ and ∂D is its boundary $u(x,y,t)$ and $v(x,y,t)$ are the velocity components to be determined, f, g, f_1 and g_1 are known functions and Re is the Reynolds number.

Which is a fundamental partial differential equation in fluid mechanics and it occurs in various areas of applied mathematics, such as modeling of gas dynamics, heat conduction, and acoustic waves (Hongqing, Huazhong, & Meiyu, 2010)

OBJECTIVE

We develop a pure Crank-Nicholson (CN) scheme and hybrid scheme of Crank-Nicholson and Du Fort & Frankel (CN-DF) from operator splitting to solve 2-D Burgers equations.

METHODOLOGY

The methodology involved investigation to the solutions of the pure Crank-Nicholson and hybrid method resulting from the Crank-Nicholson and Du-Fort & Frankel' finite difference methods resulting from operator splitting. The Crank-Nicholson method is the parent. Finally, the finite difference schemes developed were compared with those of the available analytic solutions and were be analyzed for stability.

OVERVIEW OF OPERATOR SPLITTING

Consider the Taylor's expansion

$$u(x,y,t+k) \;=\; u(x,y,t) + k\frac{\partial}{\partial t}u(x,y,t) + \frac{k^2}{2!}\frac{\partial^2}{\partial t^2}u(x,y,t) + \cdots$$

$$= \left(1 + k\frac{\partial}{\partial t} + \frac{k^2}{2!}\frac{\partial^2}{\partial t^2} + \cdots\right) u(x,y,t) \qquad = e^{k\frac{\partial}{\partial t}}\big(u(x,y,t)\big) \quad (2.4)$$

In equation (2.4) we can replace $\frac{\partial}{\partial t}$ by L that is

$$u(x,y,t+k) = e^{kL}\, u(x,y,t) \qquad (2.5)$$

The exact solution of the equation (2.1) at the grid point ($x = mh, y = lq, t = nk$) is $u(x,y,t)$ with h, q and k being the grid spacing in the x- direction, y- direction and t- direction respectively. m, l and k are intergers. $m = l = n = 0$ is the origin. The approximate solution at this point is denoted by $U_{m,l,n}$. We can then write the finite difference (FD) approximation of equation (2.5) as:

$$U_{m,l,n+1} = e^{kL}\, U_{m,l,n} \qquad (2.6)$$

OVERVIEW OF OPERATOR SPLITTING CONT…..

In equations (2.5) and (2.6) e^{kL} is called the solution operator for equation (2.1) L is replaced by finite difference approximation. In equation (2.6) L can be taken to be a sum of differential operators with respect to x.

If $L = L_1 + L_2 + L_3 + \cdots + L_s = \sum_{i=1}^{S} L_i$

Then equation (2.6) can be written as

$$U_{m,l,n+1} = e^{k\sum_{i=1}^{S} L_i}\, U_{m,l,n}$$

$$= e^{k(L_1 + L_2 + L_3 + \cdots + L_s)}\, U_{m,l,n} \qquad (2.7)$$

$$= \prod_{i=1}^{S} e^{k^{L_i}}\, U_{m,l,n} \qquad (2.9)$$

OVERVIEW OF OPERATOR SPLITTING CONT.....

The approximate solution can be obtained from equation (2.8) by first solving

$$U_{m,l,n+1}^{(s)} = e^{kL_s} U_{m,l,n} \qquad (2.10)$$

and then using this solution we can find

$$U_{m,l,n+1}^{(s-1)} = e^{kL_{s-1}} U_{m,l,n} \qquad (2.11)$$

We go on like this until we attain

$$U_{m,l,n+1}^{(1)} \qquad (2.12)$$

which is actually the approximate solution of equation (2.1)

PURE CRANK-NICHOLSON (CN) SCHEME

We consider the 2-D Burgers equation of the form

$$f_1(u,v) = -uu_x - vu_y + \frac{1}{Re}\left(u_{xx} + u_{yy}\right), (0 \le x, y \le 1) \times (0 \le t \le T) \quad (2.1.1)$$

$$f_2(u,v) = -uv_x - vv_y + \frac{1}{Re}\left(v_{xx} + v_{yy}\right), (0 \le x, y \le 1) \times (0 \le t \le T) \quad (2.1.2)$$

Here s=2

And so

$$L = L_1 + L_2 + L_3 + L_4$$

Let

$$L_1 = -\frac{1}{h} u_{m,l,n} \delta_x$$

$$L_2 = -\frac{1}{q} v_{m,l,n} \delta_y$$

$$L_3 = \frac{1}{Reh^2} {\delta_x}^2$$

$$L_4 = \frac{1}{Req^2} {\delta_y}^2$$

From equation (2.1.7) the approximate solution is found by

PURE CRANK-NICHOLSON (CN) SCHEME CONT…

$$U_{m,l,n+1} =$$

$$(e^{kL_1}\,(e^{kL_2}(e^{kL_3}(e^{kL_4}\,U_{m,l,n})))) \tag{2.1.3}$$

$$U_{m,l,n+1} =$$

$$\left(1+kL_1+\frac{1}{2}k^2L^2{}_1+..\right)\left(1+kL_2+\frac{1}{2}k^2L^2{}_2+..\right)\left(1+kL_3+\frac{1}{2}k^2L^2{}_3+..\right)\left(1+kL_4+\frac{1}{2}k^2L^2{}_4+..\right)$$

$$\approx 1+\ kL_1+kL_2+kL_3+kL_4+L_1L_2k^2+L_1L_3k^2+L_1L_4k^2+L_2L_3k^2+L_2L_4k^2+L_3L_4k^2+$$

$$\frac{1}{2}kL_1{}^2+\frac{1}{2}kL_2{}^2+\frac{1}{2}kL_3{}^2+\frac{1}{2}kL_4{}^2 \tag{2.1.4}$$

It is necessary that we first develop the pure Crank-Nicholson method resulting from this splitting. This is because other hybrid methods are derived from it. Thus the Crank-Nicholson method is as follows:

PURE CRANK-NICHOLSON (CN) SCHEME CONT…

$$L_1 U_{m,l,n} = -\frac{1}{2h} U_{m,l,n} \delta_x (U_{m,l,n} + U_{m,l,n+1}) \qquad (2.1.5)$$

$$L_2 V_{m,l,n} = -\frac{1}{2q} V_{m,l,n} \delta_y (U_{m,l,n} + U_{m,l,n+1}) \qquad (2.1.6)$$

$$L_3 U_{m,l,n} = \frac{1}{4Re\, h^2} \delta_x^2 (U_{m,l,n} + U_{m,l,n+1}) \qquad (2.1.7)$$

$$L_4 U_{m,l,n} = \frac{1}{4Re\, q^2} \delta_y^2 (U_{m,l,n} + U_{m,l,n+1}) \qquad (2.1.8)$$

$$L_1 L_2 V_{m,l,n} = \frac{1}{4hq} U_{m,l,n} V_{m,l,n} \delta_x (\delta_y (U_{m,l,n} + U_{m,l,n+1})) \qquad (2.1.9)$$

$$L_1 L_3 U_{m,l,n} = -\frac{1}{8h^3 Re} U_{m,l,n} \delta_x^3 (U_{m,l,n} + U_{m,l,n+1}) \qquad (2.1.10)$$

$$L_1 L_4 U_{m,l,n} = -\frac{1}{8h q^2 Re} U_{m,l,n} \delta_x (\delta_y (\delta_y (U_{m,l,n} + U_{m,l,n+1}))) \qquad (2.1.11)$$

$$L_2 L_3 U_{m,l,n} = -\frac{1}{8q h^2 Re} V_{m,l,n} \delta_y (\delta_x^2 (U_{m,l,n} + U_{m,l,n+1})) \qquad (2.1.12)$$

$$L_2 L_4 V_{m,l,n} = -\frac{1}{8q^3 Re} V_{m,l,n} \delta_y^3 (U_{m,l,n} + U_{m,l,n+1}) \qquad (2.1.13)$$

PURE CRANK-NICHOLSON (CN) SCHEME CONT…

$$L_3 L_4 V_{m,l,n} = -\frac{1}{8q^3 Re} V_{m,l,n} \delta_x^2 \delta_y^2 (U_{m,l,n} + U_{m,l,n+1}) \quad (2.1.14)$$

$$L_1^2 U_{m,l,n} = \frac{1}{4h^2} U_{m,l,n}^2 \delta_x^2 (U_{m,l,n} + U_{m,l,n+1}) \quad (2.1.15)$$

$$L_2^2 V_{m,l,n} = \frac{1}{4q^2} V_{m,l,n}^2 \delta_y^2 (U_{m,l,n} + U_{m,l,n+1}) \quad (2.1.16)$$

$$L_3^2 U_{m,l,n} = \frac{1}{16Re^2 h^4} \delta_x^4 (U_{m,l,n} + U_{m,l,n+1}) \quad (2.1.17)$$

$$L_4^2 U_{m,l,n} = \frac{1}{16Re^2 q^4} \delta_y^4 (U_{m,l,n} + U_{m,l,n+1}) \quad (2.1.18)$$

Using equations (2.1.5)-(2.1.18) in equation (2.1.4) and letting $q = h$, we obtain a discretization scheme by operator splitting.

29/07/2015

APPROXIMATION AT THE BOUNDARY

We use work developed by Kweyu (Kweyu , Manyonge, Koross A. , & Ssema, 2012) for the initial and boundary conditions. The solution are given as:

$$u(x,y,t) = \frac{-2y-2\pi e^{-\frac{2\pi^2 t}{Re}}\left((\cos\pi x-\sin\pi x)\sin\pi y\right)}{Re\left(100+xy+e^{-\frac{2\pi^2 t}{Re}}\left((\cos\pi x-\sin\pi x)\sin\pi y\right)\right)} \quad (2.1.19)$$

$$v(x,y,t) = \frac{-2x-2\pi e^{-\frac{2\pi^2 t}{Re}}\left((\cos\pi x-\sin\pi x)\sin\pi y\right)}{Re\left(100+xy+e^{-\frac{2\pi^2 t}{Re}}\left((\cos\pi x-\sin\pi x)\sin\pi y\right)\right)} \quad (2.1.20)$$

CRANK-NICHOLSON-DU FORT-FRANKEL (CN-DF) SCHEME

The Scheme is obtained by replacing $U_{m,l,n}$ in the pure Crank-Nicholson Scheme by

$$\frac{1}{2}(U_{m,l,n+1} + U_{m,l,n-1})$$

.

RESULTS OF THE NUMERICAL SCHEMES DEVELOPED

We present the results using the following data: $k = 0.001, h = 0.1, l = 0.1$. We now present the results. We shall display these results using tables and 3-D figures.

Table 1: Numerical Solution of u for Coupled Burgers' equation at $t = 1.0$, $y = 1.0$ and Re=5000

x	Exact Solution u (*e-006)	Pure CN u (*e-006)	Hybrid CN-DF u (*e-006)
0.1	-0.361650734301019	-0.361787155384327	-0.361653771182265
0.2	-0.725321995372639	-0.725588362525353	-0.725327925052115
0.3	-1.090398562803820	-1.090790321153970	-1.090407283922360
0.4	-1.455935883797210	-1.456451500936150	-1.455947362252530
0.5	-1.820803748461820	-1.821445349043890	-1.820818031533110
0.6	-2.183867594804460	-2.184640978240830	-2.183884811546400
0.7	-2.544179042412540	-2.545093045825040	-2.544199389476210
0.8	-2.901144228647610	-2.902209524822090	-2.901167943526820
0.9	-3.254642313537360	-3.255869840866490	-3.254669639626780
1	-3.605076050695350	-3.606475329816040	-3.605107199831220

RESULTS CONT…

Table 2: Numerical Solution of v for Coupled Burgers' equation at $t = 1.0,\ y = 1.0$ and Re=5000

x	Exact Solution v (*e-006)	Pure CN v (*e-006)	Hybrid CN-DF v (*e-006)
0.1	-3.972769188311190	-3.972865925156520	-3.972771341778260
0.2	-3.944170064848750	-3.944368960551150	-3.944174492525940
0.3	-3.913141335562600	-3.913451475311220	-3.913148239730720
0.4	-3.878873375785220	-3.879306517566780	-3.878883018208280
0.5	-3.840895689849290	-3.841464954144770	-3.840908362597330
0.6	-3.799126642198930	-3.799845044283970	-3.799142634968780
0.7	-3.753877164071580	-3.754756190904300	-3.753896732505020
0.8	-3.705807681840330	-3.706856112188090	-3.705831021264290
0.9	-3.655845501933210	-3.657068620610160	-3.655872729881250
1	-3.605076050695350	-3.606475329816040	-3.605107199831220

RESULTS CONT…

Table 3: Absolute errors in Numerical Solution of u for Coupled Burgers' Equation at $t=1.0$, $y=1.0$ and Re=5000

x	Pure CN u (*e-006)	Hybrid CN-DF-LF u (*e-006)
0.1	0.000136421083307969	0.000001518470782
0.2	0.000266367152713998	0.000002964898953
0.3	0.000391758350150040	0.000004360646750
0.4	0.000515617138939994	0.000005739343120
0.5	0.000641600582069968	0.000007141679460
0.6	0.000773383436369901	0.000008608544150
0.7	0.000914003412499920	0.000010173735920
0.8	0.001065296174479700	0.000011857676430
0.9	0.001227527329129790	0.000013663316080
1	0.001399279120689820	0.000015574875360

29/07/2015

RESULTS CONT…

Table 4: Absolute errors in Numerical Solution of v for Coupled Burgers' equation at $t=1.0$, $y=1.0$ and Re=5000

x	Pure CN v (*e-006)	Hybrid CN-DF v (*e-006)
0.1	0.000096736845329737	0.000002153467070
0.2	0.000198895702399948	0.000004427677190
0.3	0.000310139748620042	0.000006904168120
0.4	0.000433141781559954	0.000009642423060
0.5	0.000569264295480210	0.000012672748040
0.6	0.000718402085040371	0.000015992769850
0.7	0.000879026832719898	0.000019568433440
0.8	0.001048430347760030	0.000023339423960
0.9	0.001223118676950020	0.000027227948040
1	0.001399279120689820	0.000031149135870

RESULTS CONT…

The above table shows that the CN-DF scheme provides accurate results closer to the exact solutions as compared to the CN scheme.

RESULTS CONT…

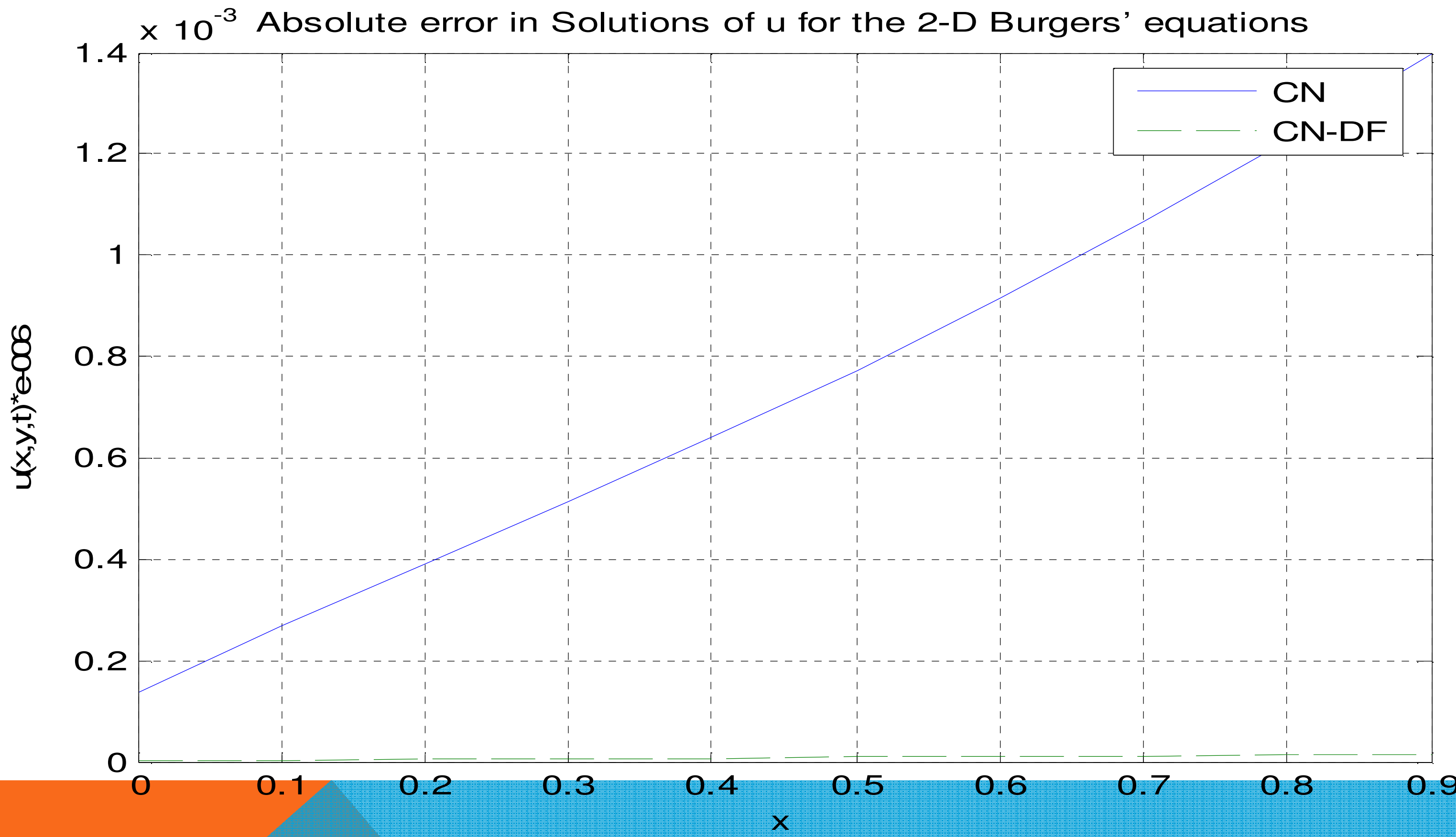


Figure 1: Absolute error in Solution of u for the 2-D Coupled Burgers' equation

29/07/2015

RESULTS CONT…

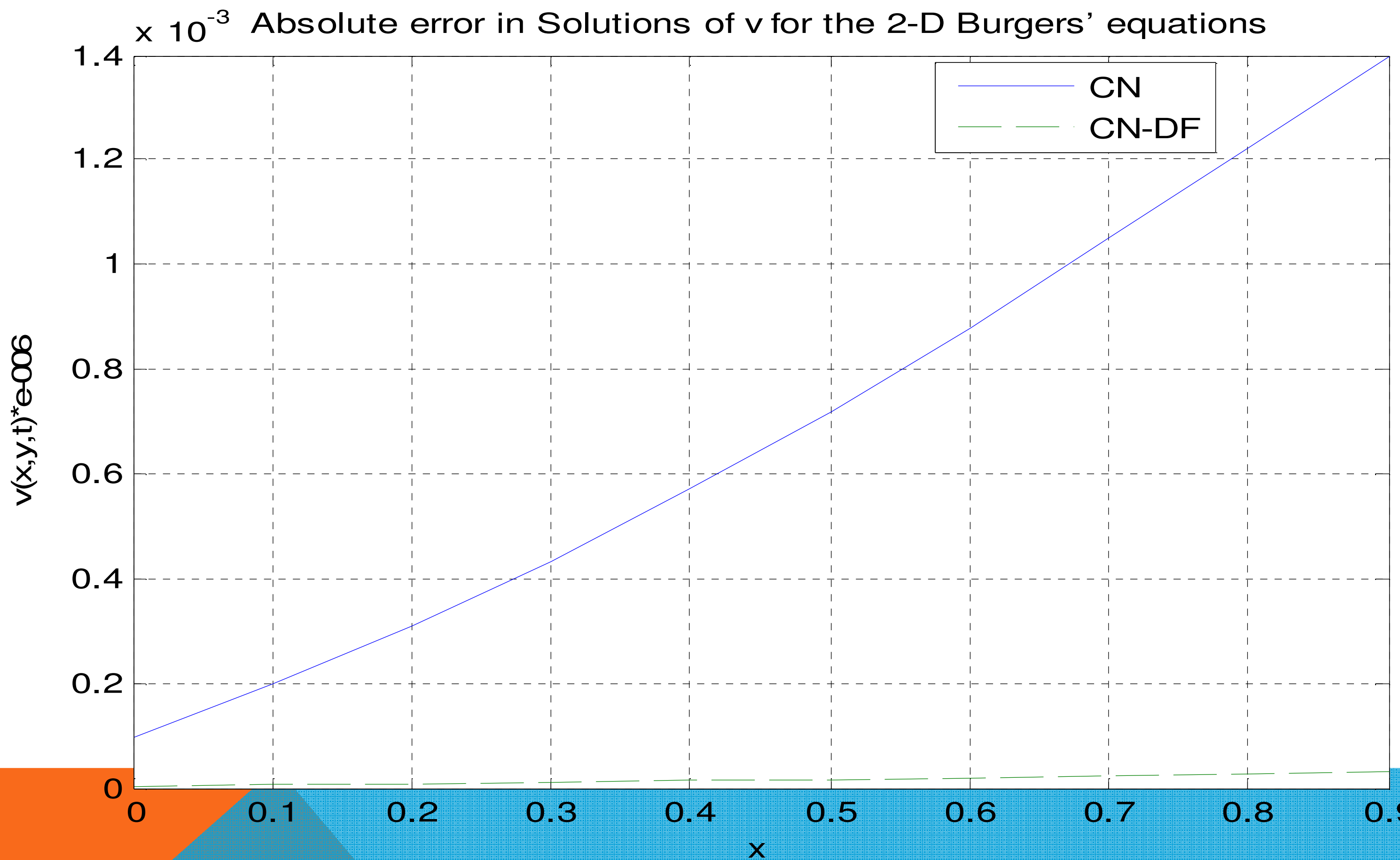


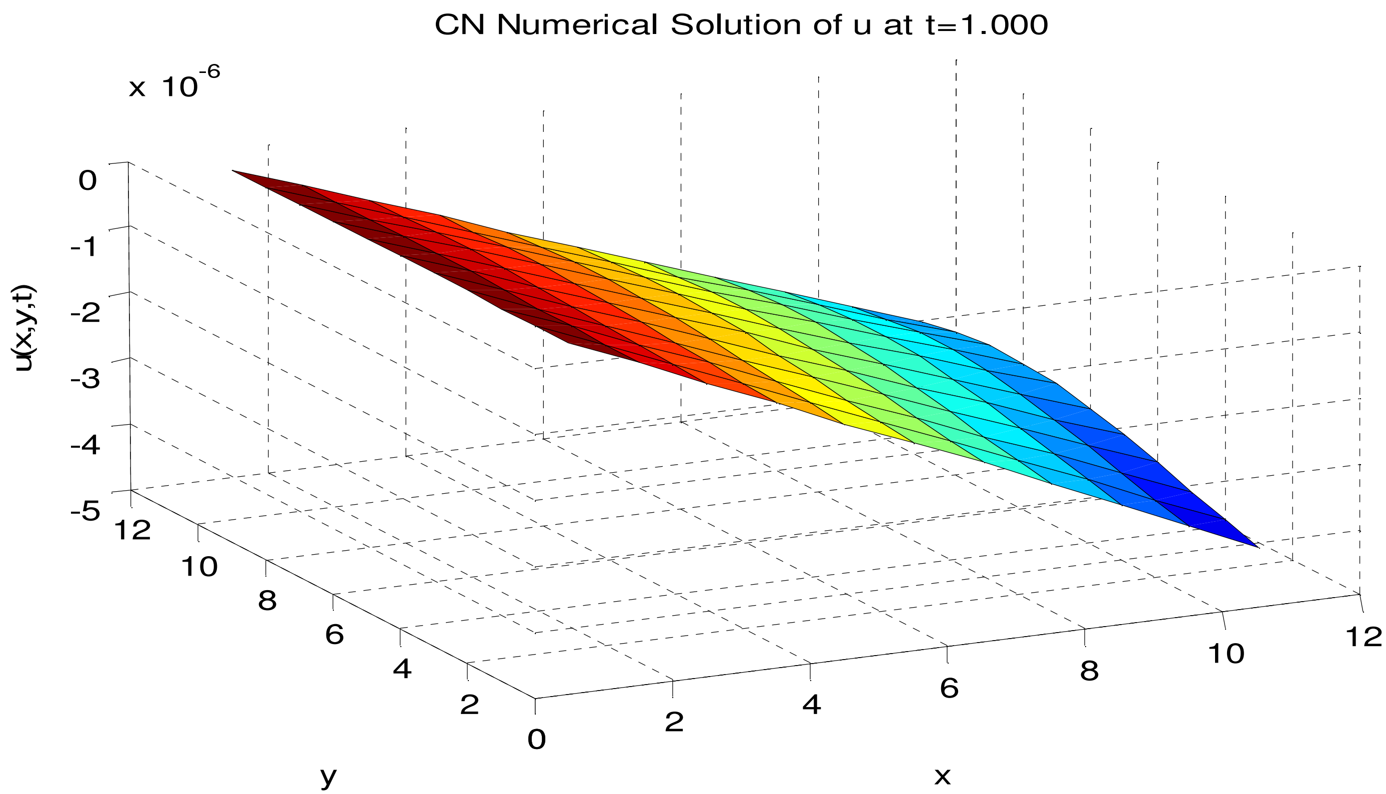
Figure 2: Absolute error in Solution of v for the 2-D Coupled Burgers' equation

RESULTS CONT…

Figure 1 & 2 shows a decreased absolute error in CN-DF compared to CN for numerical solution of both u and v.

We now present 3-D solutions:

RESULTS CONT…



29/07/2015

Figure 3: CN Numerical Solution of u at t=1.000

RESULTS CONT…

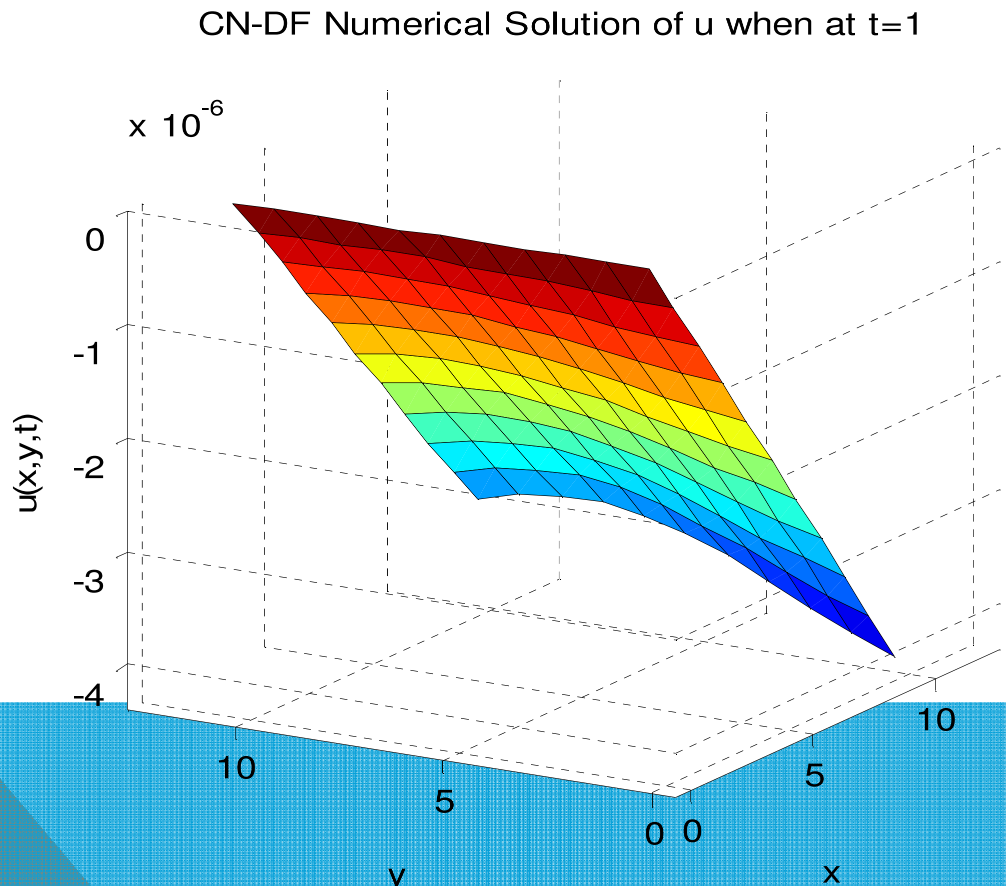


Figure 4: CN-DF Numerical Solution of u at t=1.000

RESULTS CONT…

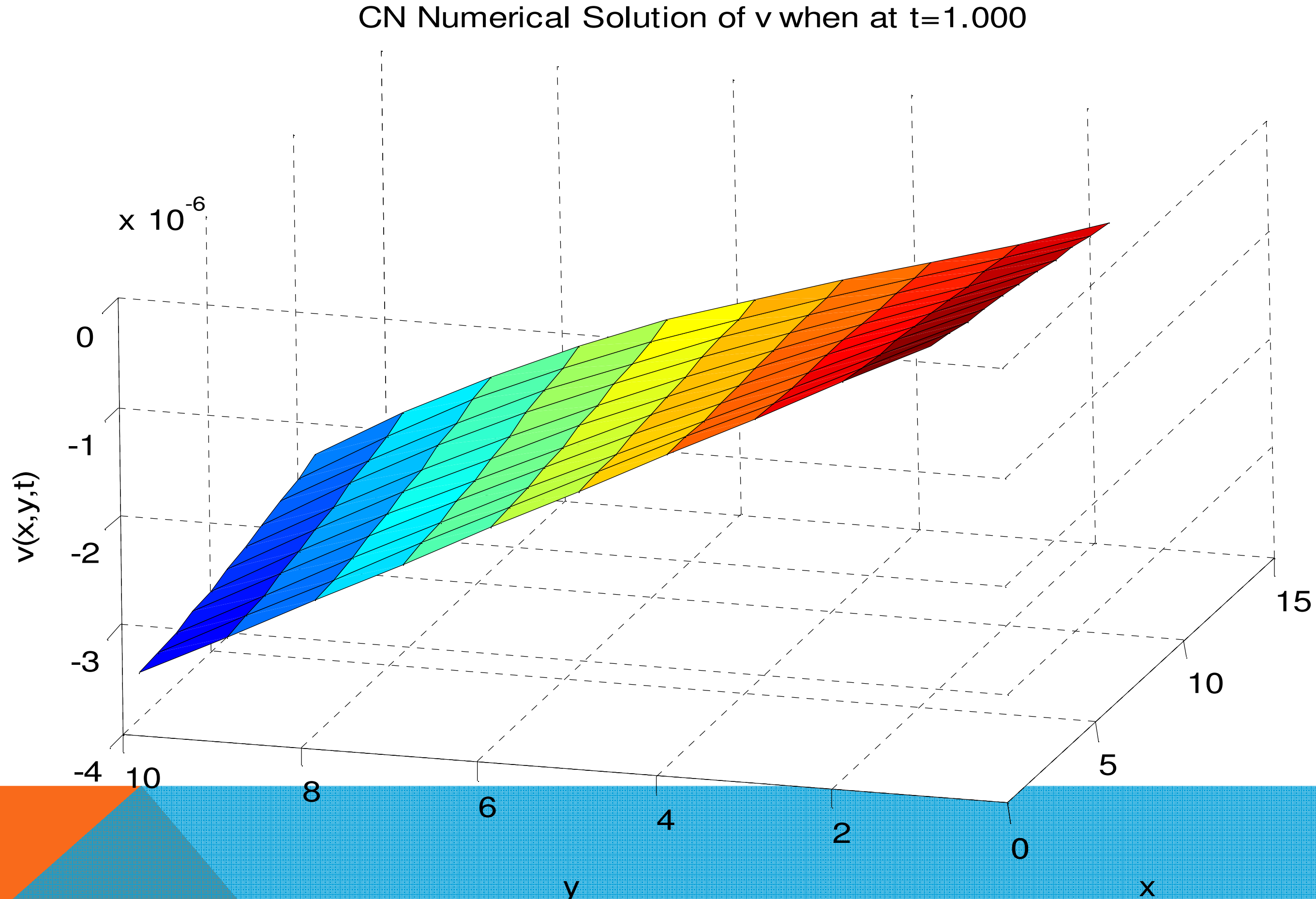


Figure 5: CN Numerical Solution of v at t=1.000

RESULTS CONT…

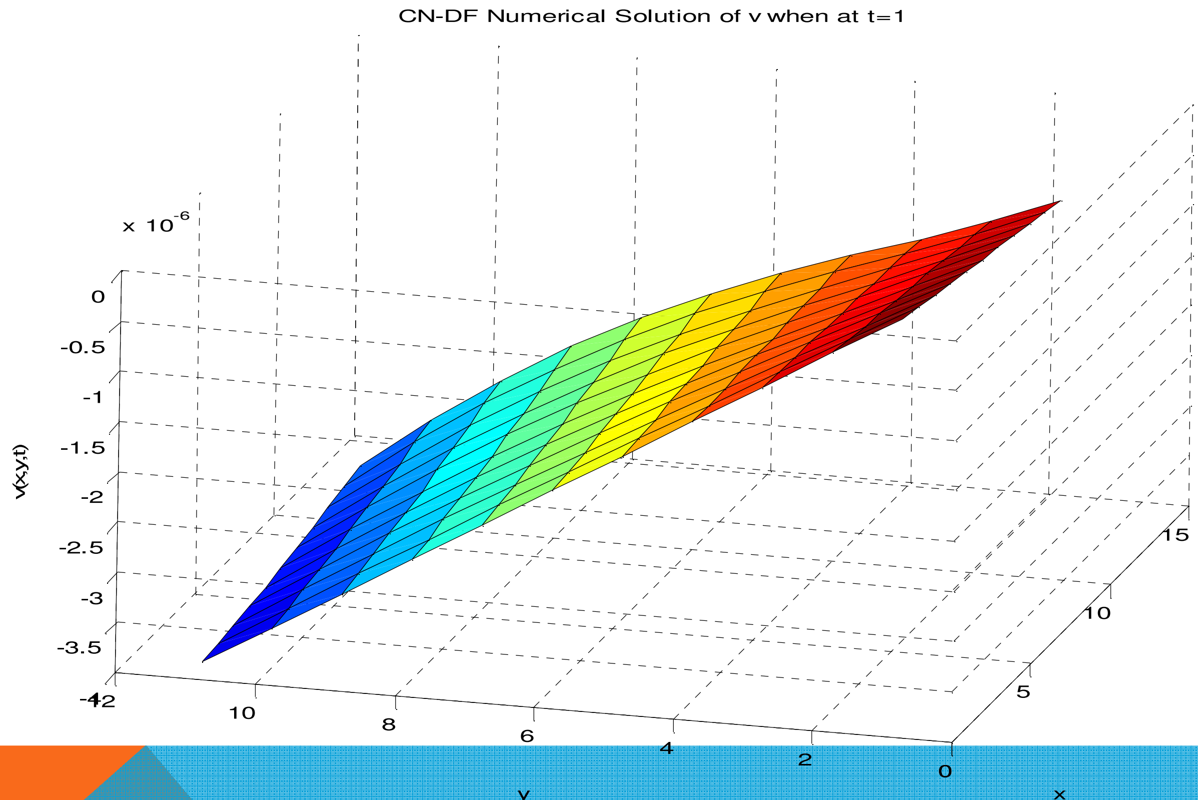


Figure 6: CN-DF Numerical Solution of v at t=1.000

CONCLUSION

We note that the 3-D solutions from all the methods developed take the same shape. It is thus established that the finite difference schemes developed are convergent.

The hybrid CN-DF scheme is the more accurate than the pure CN scheme when compared with the exact solution. The decrease in the absolute error verifies the consistency of the scheme.

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THANK YOU!!

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